

A CHARACTERIZATION OF BURNIAT SURFACES WITH $K^2 = 4$ AND OF NON NODAL TYPE

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ABSTRACT. Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 4$. Assume the bicanonical map φ of S is a morphism of degree 4 such that the image of φ is smooth. Then we prove that the surface S is a Burniat surface with $K^2 = 4$ and of non nodal type.

1. INTRODUCTION

When we consider the bicanonical map φ of a minimal surface S of general type with $p_g(S) = 0$ over the field of complex numbers, Xiao [20] gave that the image of φ is a surface if $K_S^2 \geq 2$, and Bombieri [5] and Reider [19] proved that φ is a morphism for $K_S^2 \geq 5$. In [11, 12, 14] Mendes Lopes [6] and Pardini obtained that the degree of φ is 1 for $K_S^2 = 9$; 1 or 2 for $K_S^2 = 7, 8$; 1, 2 or 4 for $K_S^2 = 5, 6$ or for $K_S^2 = 3, 4$ with a morphism φ . Moreover, there are further studies for the surface S with non birational map φ in [7, 8, 9, 11, 13, 15].

Mendes Lopes and Pardini [7] gave a characterization of a Burniat surface with $K^2 = 6$ as a minimal surface of general type with $p_g = 0$, $K^2 = 6$ and the bicanonical map of degree 4. Zhang [21] proved that a surface S is a Burniat surface with $K^2 = 5$ if the image of the bicanonical map φ of S is smooth, where a surface S is a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 5$ and the bicanonical map φ of degree 4. In this paper we extend their characterizations of Burniat surfaces with $K^2 = 6$ [7], and with $K^2 = 5$ [21] to one for the case $K^2 = 4$ as the following.

Theorem 1.1. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 4$. Assume the bicanonical map $\varphi: S \rightarrow \Sigma \subset \mathbb{P}^4$ is a morphism of degree 4 such that the image Σ of φ is smooth. Then the surface S is a Burniat surface with $K^2 = 4$ and of non nodal type.*

As we mentioned before Bombieri [5] and Reider [19] gave that the bicanonical map of a minimal surface of general type with $p_g = 0$ is a morphism for $K^2 \geq 5$. On the other hand, Mendes Lopes and Pardini [10] found that there is a family of numerical Campedelli surfaces, minimal surfaces of general type with $p_g = 0$ and $K^2 = 2$, with $\pi_1^{alg} = \mathbb{Z}_3^2$ such that the base locus of the bicanonical system consists of two points. However, we do not know whether the bicanonical system of a minimal surface of general type with $p_g = 0$ and $K^2 = 3$ or 4 has a base point or not. Thus we need to assume that the bicanonical map is a morphism in Theorem 1.1.

Bauer and Catanese [2, 3, 4] studied Burniat surfaces with $K^2 = 4$. Let S be a Burniat surface with $K^2 = 4$. When S is of non nodal type it has the ample canonical divisor, but when S is of nodal type it has one (-2) -curve. For the case

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of nodal type we will discuss to characterize Burniat surfaces with $K^2 = 4$ and of nodal type in the future article.

We follow and use the strategies of Mendes Lopes and Pardini [7], and of Zhang [21] as main tools of this article. The paper is organized as follows: in Section 3 we recall some useful formulas and Propositions for a double cover from [7], and we give a description of a Burniat surface with $K^2 = 4$ and of non nodal type; in Section 4 we analyze branch divisors of the bicanonical morphism φ of degree 4 of a minimal surface of general type with $p_g = 0$ and $K^2 = 4$ when the image of φ is smooth; in Section 5 we give a proof of Theorem 1.1.

2. NOTATION AND CONVENTIONS

In this section we fix the notation which will be used in the paper. We work over the field of complex numbers.

Let X be a smooth projective surface. Let Γ be a curve in X and $\tilde{\Gamma}$ be the normalization of Γ . We set:

K_X : the canonical divisor of X ;
 $q(X)$: the irregularity of X , that is, $h^1(X, \mathcal{O}_X)$;
 $p_g(X)$: the geometric genus of X , that is, $h^0(X, \mathcal{O}_X(K_X))$;
 $p_g(\Gamma)$: the geometric genus of Γ , that is, $h^0(\tilde{\Gamma}, \mathcal{O}_{\tilde{\Gamma}}(K_{\tilde{\Gamma}}))$;
 $\chi_{top}(X)$: the topological Euler characteristic of X ;
 $\chi(\mathcal{F})$: the Euler characteristic of a sheaf \mathcal{F} on X , that is, $\sum_{i=0}^2 (-1)^i h^i(X, \mathcal{F})$;
 \equiv : the linear equivalence of divisors on a surface;
 $(-n)$ -curve: a smooth irreducible rational curve with the self-intersection number $-n$, in particular we call that a (-1) -curve is exceptional and a (-2) -curve is nodal;
We usually omit the sign \cdot of the intersection product of two divisors on a surface. And we do not distinguish between line bundles and divisors on a smooth variety.

3. PRELIMINARIES

3.1. Double covers. Let S be a smooth surface and $B \subset S$ be a smooth curve (possibly empty) such that $2L \equiv B$ for a line bundle L on S . Then there exists a double cover $\pi: Y \rightarrow S$ branched over B . We get

$$\pi_* \mathcal{O}_Y = \mathcal{O}_S \oplus L^{-1},$$

and the invariants of Y from ones of S as follows:

$$K_Y^2 = 2(K_S + L)^2, \quad \chi(\mathcal{O}_Y) = 2\chi(\mathcal{O}_S) + \frac{1}{2}L(K_S + L),$$

$$p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(K_S + L)),$$

$$q(Y) = q(S) + h^1(S, \mathcal{O}_S(K_S + L)).$$

We begin with the following Proposition in [7].

Proposition 3.1 (Proposition 2.1 in [7]). *Let S be a smooth surface with $p_g(S) = q(S) = 0$, and let $\pi: Y \rightarrow S$ be a smooth double cover. Suppose that $q(Y) > 0$. Denote the Albanese map of Y by $\alpha: Y \rightarrow A$. Then*

- (i) *the Albanese image of Y is a curve C ;*
- (ii) *there exist a fibration $g: S \rightarrow \mathbb{P}^1$ and a degree 2 map $p: C \rightarrow \mathbb{P}^1$ such that $p \circ \alpha = g \circ \pi$.*

Proposition 3.2 (Corollary 2.2 in [7]). *Let S be a smooth surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 \geq 3$, and let $\pi: Y \rightarrow S$ be a smooth double cover. Then $K_Y^2 \geq 16(q(Y) - 1)$.*

3.2. Bidouble covers. Let Y be a smooth surface and $D_i \subset Y$, $i = 1, 2, 3$ be smooth divisors such that $D := D_1 + D_2 + D_3$ is a normal crossing divisor, $2L_1 \equiv D_2 + D_3$ and $2L_2 \equiv D_1 + D_3$ for line bundles L_1, L_2 on Y . By [18] there exists a bidouble cover $\psi: \bar{Y} \rightarrow Y$ branched over D . We obtain

$$\psi_* \mathcal{O}_{\bar{Y}} = \mathcal{O}_Y \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1},$$

where $L_3 = L_1 + L_2 - D_3$.

We describe a Burniat surface with $K^2 = 4$ and of non nodal type [2].

Notation 3.3. Let $\rho: \Sigma \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 at 5 points p_1, p_2, p_3, p_4, p_5 in general position. We denote that l is the pull-back of a line in \mathbb{P}^2 , e_i is the exceptional curve over p_i , $i = 1, 2, 3, 4, 5$, and e'_i is the strict transform of the line joining p_j and p_k , $\{i, j, k\} = \{1, 2, 3\}$. Also, g_i (resp. h_i) denotes the strict transform of the line joining p_4 (resp. p_5) and p_i , $i = 1, 2, 3$. Then the picard group of Σ is generated by l, e_1, e_2, e_3, e_4 and e_5 . We get that $-K_\Sigma \equiv 3l - \sum_{i=1}^5 e_i$ is very ample. The surface Σ is embedded by the linear system $| -K_\Sigma |$ as a smooth surface of degree 4 in \mathbb{P}^4 , called a del Pezzo surface of degree 4.

We consider smooth divisors

$$B_1 = e_1 + e'_1 + g_2 + h_2 \equiv 3l + e_1 - 3e_2 - e_3 - e_4 - e_5,$$

$$B_2 = e_2 + e'_2 + g_3 + h_3 \equiv 3l - e_1 + e_2 - 3e_3 - e_4 - e_5, \text{ and}$$

$$B_3 = e_3 + e'_3 + g_1 + h_1 \equiv 3l - 3e_1 - e_2 + e_3 - e_4 - e_5.$$

Then $B := B_1 + B_2 + B_3$ is a normal crossing divisor, $2L'_1 \equiv B_2 + B_3$ and $2L'_2 \equiv B_1 + B_3$ for line bundles L'_1, L'_2 on Σ . We obtain a bidouble cover $\varphi: S \rightarrow \Sigma \subset \mathbb{P}^4$. We remark that the example is a minimal surface S of general type with $p_g(S) = 0$, $K_S^2 = 4$ and the bicanonical morphism φ of degree 4 having the ample K_S .

4. BRANCH DIVISORS OF THE BICANONICAL MAP

Notation 4.1. Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 4$. Assume that the bicanonical map φ of S is a morphism of degree 4 and the image Σ of φ is smooth in \mathbb{P}^4 . By [17] Σ is a del Pezzo surface of degree 4 in Notation 3.3. We denote $\rho, l, e_i, e'_j, g_j, h_j$, $i = 1, 2, 3, 4, 5$, $j = 1, 2, 3$ as the notations in Notation 3.3. Denote $\gamma \equiv l - e_4 - e_5$, $\delta \equiv 2l - \sum_{i=1}^5 e_i$, $f_i \equiv l - e_i$ and $F_i \equiv \varphi^*(f_i)$ for $i = 1, 2, 3, 4, 5$.

We follow the strategies of [7, 21]. We start with the following proposition similar to one in Section 4 of [21].

Proposition 4.2 (Note Proposition 4.2 in [21]). *For $i = 1, 2, 3, 4, 5$ if $f_i \in |f_i|$ is general, then $\varphi^*(f_i)$ is connected, hence $|F_i|$ induces a genus 3 fibration $u_i: S \rightarrow \mathbb{P}^1$.*

Proof. We get a similar proof from Proposition 4.2 in [21]. \square

Proposition 4.3 (Note Proposition 4.4 in [7] and Proposition 4.3 in [21]). *The bicanonical morphism φ is finite, the canonical divisor K_S is ample, and for $i = 1, 2, 3, 4, 5$, the pull-back of an irreducible curve in $|f_i|$ is also irreducible (possibly non-reduced).*

Proof. Noether's formula gives $\chi_{top}(S) = 8$ by $\chi(\mathcal{O}_S) = 1$ and $K_S^2 = 4$. Then we get $h^2(S, \mathbb{Q}) = h^2(\Sigma, \mathbb{Q}) = 6$ by $p_g(S) = q(S) = 0$. So $\varphi^*: H^2(\Sigma, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$ is an isomorphism preserving the intersection form up to multiplication by 4. Therefore φ is finite and K_S is ample.

For an irreducible curve $f_1 \in |f_1|$, if $\varphi^*(f_1)$ is reducible, then it contains an irreducible component C with $C^2 < 0$. Put $D = C - \frac{C \cdot \varphi^*(e_1)}{4} \varphi^*(f_1)$. Then $D^2 =$

$C^2 < 0$, and $D\varphi^*(e_1) = 0$. And $\left(C - \frac{C\varphi^*(e_1)}{4}\varphi^*(f_1)\right)\varphi^*(e_i) = 0$ for $i = 2, 3, 4, 5$ since e_i is contained in one fiber of the pencil $|f_1|$. We obtain that the intersection matrix of $\varphi^*(l)$, $C - \frac{C\varphi^*(e_1)}{4}\varphi^*(f_1)$, $\varphi^*(e_i)$, $i = 1, 2, 3, 4, 5$ has rank 7. But it is a contradiction because $h^2(S, \mathbb{Q}) = 6$. Thus $\varphi^*(f_1)$ is irreducible. We can similarly prove the other cases. \square

Lemma 4.4 (Lemma 4.4 in [21]). *Let $\phi: T' \rightarrow T$ be a finite morphism between two smooth surfaces. Let h be a divisor on T such that $|\phi^*(h)| = \phi^*(|h|)$. Let M be a divisor on T' such that the linear system $|M|$ has no fixed part. Suppose that $\phi^*(h) - M$ is effective. Then there exists a divisor $m \subset T$ such that $|M| = \phi^*(|m|)$. Furthermore the line bundle $h - m$ is effective.*

Lemma 4.5 (Note Lemma 4.5 in [21]). *There does not exist a divisor d on Σ such that $h^0(\Sigma, \mathcal{O}_\Sigma(d)) > 1$ and that the line bundle $-K_\Sigma - 2d$ is effective.*

Proof. Suppose that there exists such a divisor d . Assume $d \equiv al - \sum_{i=1}^5 b_i e_i$ for some integers a, b_i , $i = 1, 2, 3, 4, 5$. Then $a \leq 1$ because $-K_\Sigma - 2d$ is effective. On the other hand, $a \geq 1$ by the condition $h^0(\Sigma, \mathcal{O}_\Sigma(d)) > 1$. Thus $a = 1$, and at most one of b_1, \dots, b_5 is positive. Then the line bundle $-K_\Sigma - 2d \equiv l - \sum_{i=1}^5 (1 - 2b_i)e_i$ cannot be effective since there is no line on \mathbb{P}^2 passing through 3 points in general position. \square

We prove the following lemma as one of Lemma 4.6 in [21] since we have Lemmas 4.4 and 4.5.

Lemma 4.6 (Note Lemma 4.6 in [21]). *Let $D \subset S$ be a divisor. If there exists a divisor d on Σ such that*

- (i) $\varphi^*(d) \equiv 2D$;
 - (ii) *the line bundle $-K_\Sigma - d$ is effective,*
- then $h^0(S, \mathcal{O}_S(D)) \leq 1$.*

Proof. Suppose $h^0(S, \mathcal{O}_S(D)) > 1$. We may write $|D| = |M| + F$ where $|M|$ is the moving part and F is the fixed part. Since $|2K_S| = |\varphi^*(-K_\Sigma)| = \varphi^*(|-K_\Sigma|)$ and $\varphi^*(-K_\Sigma) - M > \varphi^*(-K_\Sigma - d)$ is effective, there is a divisor m on Σ such that $\varphi^*(|m|) = |M|$ by Lemma 4.4. Choose an element $M_1 \in |M|$ and an effective divisor N on S such that $2M_1 + 2F + N \equiv \varphi^*(-K_\Sigma)$. We find $h \in |-K_\Sigma|$ and $m_1 \in |m|$ such that $2M_1 + 2F + N = \varphi^*(h)$ and $2M_1 = \varphi^*(2m_1)$. Thus we conclude that $h - 2m_1$ is effective. It is a contradiction by Lemma 4.5. \square

Now we investigate the pull-backs of (-1) -curves on the surface Σ via the bi-canonical morphism $\varphi: S \rightarrow \Sigma \subset \mathbb{P}^4$. There are sixteen (-1) -curves on Σ which are $e_i, e'_j, g_j, h_j, \gamma$ and δ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$.

Lemma 4.7 (Lemma 5.1 in [7]). *Let $C \subset \Sigma$ be a (-1) -curve. Then we have either*

- (i) $\varphi^*(C)$ *is a reduced smooth rational (-4) -curve; or*
- (ii) $\varphi^*(C) = 2E$ *where E is an irreducible curve with $E^2 = -1$, $K_S E = 1$.*

Lemma 4.8. *There are at most three disjoint (-4) -curves on S .*

Proof. Let r be the cardinality of a set of smooth disjoint rational curves with self-intersection number -4 on S . Then

$$\frac{25}{12}r \leq c_2(S) - \frac{1}{3}K_S^2 = \frac{20}{3}$$

by [16] which is $r \leq 3$. \square

Remark 4.9. We consider an exceptional curve e on Σ which is different from δ and is not an ρ -exceptional curve (i.e. $\rho(e)$ is not a point in \mathbb{P}^2). Then we can find an automorphism τ on Σ induced by a Cremona transformation with respect to 3 points among 5 points p_1, p_2, p_3, p_4, p_5 in general position on \mathbb{P}^2 such that an exceptional curve $\tau(e)$ on Σ is different from δ and is an ρ -exceptional curve.

Proposition 4.10. *There exist at least two disjoint (-1) -curves different from δ on Σ such that those pull-backs by the bicanonical morphism φ are (-4) -curves.*

Proof. Let R be the ramification divisor of the bicanonical morphism $\varphi: S \rightarrow \Sigma \subset \mathbb{P}^4$. By Hurwitz formula $K_S \equiv \varphi^*(K_\Sigma) + R$, we get $R \equiv K_S + \varphi^*(-K_\Sigma) \equiv 3K_S$. Because $\varphi^*(-K_\Sigma) \equiv 2K_S$ since the image Σ of φ is a del Pezzo surface of degree 4 in \mathbb{P}^4 (Note Notations 3.3 and 4.1).

We assume $\varphi^*(e_i) = 2E_i$, $\varphi^*(e'_j) = 2E'_j$, $\varphi^*(g_j) = 2G_j$, $\varphi^*(h_j) = 2H_j$ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$, and $\varphi^*(\gamma) = 2\Gamma$. Put

$$R_1 := R - \left(\sum_{i=1}^3 (E_i + E'_i + G_i + H_i) + E_4 + E_5 + \Gamma \right).$$

It implies $2R_1 \equiv \varphi^*(-l)$. By the assumption, φ is ramified along reduced curves E_i, E'_j, G_j, H_j for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$, and Γ . So R_1 is a nonzero effective divisor. But it is a contradiction because $0 < (2R_1)(2K_S) = \varphi^*(-l)\varphi^*(-K_\Sigma) < 0$ since φ is finite and K_S is ample by Proposition 4.3. Thus by Lemma 4.7 and Remark 4.9 we may consider $\varphi^*(e_5) = E_5$, where E_5 is a reduced smooth rational (-4) -curve.

Again, we assume $\varphi^*(e_i) = 2E_i$, $\varphi^*(e'_j) = 2E'_j$, $\varphi^*(g_j) = 2G_j$, $\varphi^*(h_j) = 2H_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$, and $\varphi^*(\gamma) = 2\Gamma$. Put

$$R_2 := R - \left(\sum_{i=1}^3 (E_i + E'_i + G_i + H_i) + E_4 + \Gamma \right).$$

It induces $2R_2 \equiv \varphi^*(-l + e_5)$. Then the nonzero divisor R_2 is effective. Because φ is ramified along reduced curves E_i, E'_j, G_j, H_j for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$, and Γ from the assumption. It gives a contradiction because $0 < (2R_2)(2K_S) = \varphi^*(-l + e_5)\varphi^*(-K_\Sigma) < 0$ since φ is finite and K_S is ample by Proposition 4.3. By Lemma 4.7 we get an (-1) -curve e with an (-4) -curve $\varphi^*(e)$ among e_i, e'_j, g_j, h_j for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$, and γ .

We have two (-1) -curves e and e_5 different from δ on Σ such that $\varphi^*(e)$ and $\varphi^*(e_5)$ are (-4) -curves on S . We verify that e and e_5 are disjoint. By Remark 4.9 we consider that the (-1) -curve e is γ . It is enough to assume $\varphi^*(e_i) = 2E_i$, $\varphi^*(e'_j) = 2E'_j$, $\varphi^*(g_j) = 2G_j$, $\varphi^*(h_j) = 2H_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. Then put

$$R_3 := R - \left(\sum_{i=1}^3 (E_i + E'_i + G_i + H_i) + E_4 \right).$$

We get $2R_3 \equiv \varphi^*(-e_4)$. The nonzero divisor R_3 is effective. Because φ is ramified along reduced curves E_i, E'_j, G_j, H_j for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ from the assumption. It contradicts because $0 < (2R_3)(2K_S) = \varphi^*(-e_4)\varphi^*(-K_\Sigma) < 0$ since φ is finite and K_S is ample by Proposition 4.3. \square

Proposition 4.11. *There do not exist three (-1) -curves C_1, C_2 and C_3 different from δ on Σ satisfying*

- (i) $C_i \cap C_j = \emptyset$ for distinct $i, j \in \{1, 2, 3\}$;
- (ii) $\varphi^*(C_i)$ for $i = 1, 2, 3$ are (-4) -curves.

Proof. Assume that the proposition is not true. We may consider $C_1 = e_2$, $C_2 = e_4$ and $C_3 = e_5$ by Remark 4.9. Then $E_2 = \varphi^*(e_2)$, $E_4 = \varphi^*(e_4)$ and $E_5 = \varphi^*(e_5)$ are reduced smooth rational (-4) -curves. And $\varphi^*(e'_2) = 2E'_2$ with $E_2'^2 = -1$, $K_S E'_2 = 1$ by Lemmas 4.7 and 4.8. Then

$$\begin{aligned} 2K_S &\equiv \varphi^* \left(3l - \sum_{i=1}^5 e_i \right) \equiv \varphi^*(e'_2 + 2f_2 + e_2 - e_4 - e_5) \\ &\equiv 2E'_2 + 2F_2 + E_2 - E_4 - E_5. \end{aligned}$$

We get $2(K_S - E'_2 - F_2 + E_4 + E_5) \equiv E_2 + E_4 + E_5$. We consider a double cover $\pi: Y \rightarrow S$ branched over E_2 , E_4 and E_5 . By the formula in Subsection 3.1 we obtain

$$\begin{aligned} K_Y^2 &= 2(2K_S - E'_2 - F_2 + E_4 + E_5)^2 = 14, \\ \chi(\mathcal{O}_Y) &= 2 + \frac{(K_S - E'_2 - F_2 + E_4 + E_5) \cdot (2K_S - E'_2 - F_2 + E_4 + E_5)}{2} = 2, \\ p_g(Y) &= h^0(S, \mathcal{O}_S(2K_S - E'_2 - F_2 + E_4 + E_5)) \\ &= h^0(S, \mathcal{O}_S(\varphi^*(-K_S - e'_2 - f_2 + e_4 + e_5) + E'_2)) \\ &= h^0(S, \mathcal{O}_S(\varphi^*(l) + E'_2)) \geq 3. \end{aligned}$$

Thus we have $q(Y) \geq 2$, and so $K_Y^2 < 16(q(Y) - 1)$. It is a contradiction by Proposition 3.2. \square

Assumption 4.12. From Lemma 4.7, Propositions 4.10 and 4.11 we may assume that $\varphi^*(e_4) = E_4$ and $\varphi^*(e_5) = E_5$ by Remark 4.9, where E_4 and E_5 are (-4) -curves, $\varphi^*(e_i) = 2E_i$, $\varphi^*(e'_i) = 2E'_i$, $\varphi^*(g_j) = 2G_j$ and $\varphi^*(h_j) = 2H_j$ for $i = 1, 2, 3$ and $j = 1, 2$.

Notation 4.13. $2(E_j + E'_k)$ and $2(E'_j + E_k)$ are two double fibers of $u_i: S \rightarrow \mathbb{P}^1$ induced by $|F_i|$ where $\{i, j, k\} = \{1, 2, 3\}$. Set $\eta_i \equiv (E_j + E'_k) - (E'_j + E_k)$ where $\{i, j, k\} = \{1, 2, 3\}$, and set $\eta \equiv K_S - \sum_{i=1}^3 (E_i + E'_i)$. Then $2\eta \equiv -E_4 - E_5$, and by Lemma 8.3, Chap. III in [1] $\eta_i \neq 0$ for $i = 1, 2, 3$. It implies that η_i , $i = 1, 2, 3$ are torsions of order 2.

Proposition 4.14 (Note Proposition 5.9 (*resp.* 4.13) in [7] (*resp.* [21])). *For a general curve $F_i \in |F_i|$, $i = 1, 2, 3$,*

$$F_j|_{F_i} \equiv K_{F_i} \text{ if } i \neq j.$$

Proof. We verify that $F_2|_{F_1} \equiv K_{F_1}$. Since $2K_S \equiv F_1 + 2(2E_1 + E'_3 + E'_2) - E_4 - E_5$, we get

$$2(K_S - (2E_1 + E'_3 + E'_2) + E_4 + E_5) \equiv F_1 + E_4 + E_5.$$

It gives a double cover $\pi: Y \rightarrow S$ branched over F_1 , E_4 and E_5 . We have

$$\chi(\mathcal{O}_Y) = 3$$

and

$$\begin{aligned} p_g(Y) &= h^0(S, \mathcal{O}_S(F_1 + 2E_1 + E'_3 + E'_2)) \\ &= h^0(S, \mathcal{O}_S(\varphi^*(f_1 + e_1) + E'_3 + E'_2)) \\ &= h^0(S, \mathcal{O}_S(\varphi^*(l) + E'_3 + E'_2)) \geq 3, \end{aligned}$$

thus $q(Y) \geq 1$. By Proposition 3.1 the Albanese pencil of Y is the pull-back of a pencil $|F|$ of S such that $\pi^*(F)$ is disconnected for a general element F in $|F|$. Thus $FF_1 = 0$ because π is branched over F_1 . It means $|F| = |F_1|$. For a general

element $F_1 \in |F_1|$, $\pi^*(F_1)$ is an unramified double cover of F_1 given by the relation $2(K_S - (2E_1 + E'_3 + E'_2) + E_4 + E_5)|_{F_1}$. Since $\pi^*(F_1)$ is disconnected, we get

$$\begin{aligned} (K_S - (2E_1 + E'_3 + E'_2) + E_4 + E_5)|_{F_1} &\equiv (K_S - 2E_1)|_{F_1} \\ &\equiv (K_S - 2E_1 - 2E'_3)|_{F_1} \\ &\equiv (K_S - F_2)|_{F_1} \end{aligned}$$

is trivial. Thus $F_2|_{F_1} \equiv K_{F_1}$. \square

Lemma 4.15. *We have:*

- (i) $\chi(\mathcal{O}_S(K_S + \eta + \eta_i)) = -1$, $h^2(S, \mathcal{O}_S(K_S + \eta + \eta_i)) = 0$;
- (ii) $h^0(F_i, \mathcal{O}_{F_i}(K_{F_i} + \eta|_{F_i})) \leq 2$;
- (iii) $h^1(S, \mathcal{O}_S(\eta - \eta_i)) = 1$.

Proof. (i) By Riemann-Roch theorem, $\chi(S, \mathcal{O}_S(K_S + \eta + \eta_i)) = -1$ since $2\eta \equiv -E_4 - E_5$. Moreover, $h^0(S, \mathcal{O}_S(-\eta + \eta_i)) = 0$ because $2(-\eta + \eta_i) \equiv E_4 + E_5$ and E_4, E_5 are reduced (-4) -curves. It implies $h^2(S, \mathcal{O}_S(K_S + \eta + \eta_i)) = 0$ by Serre duality.

(ii) We may assume $i = 1$. By $\eta_1|_{F_1} \equiv \mathcal{O}_{F_1}$ we have an exact sequence

$$0 \longrightarrow \mathcal{O}_S(K_S + \eta + \eta_1) \longrightarrow \mathcal{O}_S(K_S + \eta + \eta_1 + F_1) \longrightarrow \mathcal{O}_{F_1}(K_{F_1} + \eta|_{F_1}) \longrightarrow 0.$$

Then we get

$$\begin{aligned} h^0(F_1, \mathcal{O}_{F_1}(K_{F_1} + \eta|_{F_1})) &\leq h^0(S, K_S + \eta + \eta_1 + F_1) - h^0(S, K_S + \eta + \eta_1) \\ &\quad + h^1(S, K_S + \eta + \eta_1) \\ &= h^0(S, K_S + \eta + \eta_1 + F_1) - \chi(\mathcal{O}_S(K_S + \eta + \eta_1)) \\ &\quad + h^2(S, K_S + \eta + \eta_1) \\ &= h^0(S, \mathcal{O}_S(K_S + \eta + \eta_1 + F_1)) + 1. \end{aligned}$$

Note $K_S + \eta + \eta_1 + F_1 \equiv 2K_S - (E_1 + E'_1)$. Since the linear system $|2K_S|$ embeds $E_1 + E'_1$ as a pair of skew lines in \mathbb{P}^4 , we have $h^0(S, \mathcal{O}_S(2K_S - (E_1 + E'_1))) = 1$. Hence $h^0(F_1, \mathcal{O}_{F_1}(K_{F_1} + \eta|_{F_1})) \leq 2$.

(iii) We have $2(\eta - \eta_i) \equiv -E_4 - E_5$. It implies $h^0(S, \mathcal{O}_S(\eta - \eta_i)) = 0$. Thus $-h^1(S, \mathcal{O}_S(\eta - \eta_i)) + h^2(S, \mathcal{O}_S(\eta - \eta_i)) = 1$ by Riemann-Roch theorem. We show $h^0(S, \mathcal{O}_S(K_S - \eta + \eta_1)) = 2$ by Serre duality. Indeed, since E_4, E_5 are rational (-4) -curves and $(2K_S + E_4 + E_5)(E_4 + E_5) = 0$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_S(2K_S) \longrightarrow \mathcal{O}_S(2K_S + E_4 + E_5) \longrightarrow \mathcal{O}_{E_4 \cup E_5} \longrightarrow 0.$$

The canonical divisor K_S is ample in Proposition 4.3. It follows $h^0(S, \mathcal{O}_S(2K_S)) = 5$ and $h^1(S, \mathcal{O}_S(2K_S)) = 0$ by Kodaira vanishing theorem and Riemann-Roch theorem. Thus the long cohomology sequence induces $h^0(S, \mathcal{O}_S(2K_S + E_4 + E_5)) = 7$. Moreover, since $h^0(\Sigma, \mathcal{O}_\Sigma(3l - e_1 - e_2 - e_3)) = 7$ and $2(K_S - \eta + \eta_1) \equiv 2K_S + E_4 + E_5 \equiv \varphi^*(3l - e_1 - e_2 - e_3)$, we get $|2(K_S - \eta + \eta_1)| = \varphi^*(|3l - e_1 - e_2 - e_3|)$. Also, $h^0(S, \mathcal{O}_S(K_S - \eta + \eta_1)) = h^0(S, \mathcal{O}_S(F_1 + E'_1 + E_1)) \geq 2$ because $K_S - \eta + \eta_1 \equiv F_1 + E'_1 + E_1$. We consider $|K_S - \eta + \eta_1| = |M| + F$ where $|M|$ is the moving part and F is the fixed part. By Lemma 4.4 there is a divisor m on Σ such that $|M| = \varphi^*(|m|)$. Then $3l - e_1 - e_2 - e_3 - 2m$ is effective by arguing as in the proof of Lemma 4.6. So $m \equiv f_i$ for some $i \in \{1, 2, 3\}$. Hence $h^0(S, \mathcal{O}_S(K_S - \eta + \eta_1)) = h^0(S, \mathcal{O}_S(M)) = h^0(\Sigma, \mathcal{O}_\Sigma(f_i)) = 2$. \square

Corollary 4.16 (Corollary 4.15 in [21]). *For a general curve $F_i \in |F_i|$, $i = 1, 2, 3$ we have*

$$\begin{aligned} (-\eta + \eta_j)|_{F_i} &\equiv \mathcal{O}_{F_i} \text{ if } i \neq j; \\ \eta_i|_{F_i} &\equiv \mathcal{O}_{F_i}; \quad (-\eta + \eta_i)|_{F_i} \not\equiv \mathcal{O}_{F_i}. \end{aligned}$$

Proof. By Lemma 4.14

$$\begin{aligned}\eta|_{F_1} &\equiv (K_S - (E_1 + E'_1))|_{F_1} \equiv K_{F_1} - (E_1 + E'_1)|_{F_1} \equiv (F_2 - (E_1 + E'_1))|_{F_1} \\ &\equiv (2(E_1 + E'_3) - (E_1 + E'_1))|_{F_1} \equiv (E_1 - E'_1)|_{F_1}.\end{aligned}$$

Since $\eta_2|_{F_1} \equiv \eta_3|_{F_1} \equiv (E_1 - E'_1)|_{F_1}$ we get $(-\eta + \eta_j)|_{F_i} \equiv \mathcal{O}_{F_i}$ for $i \neq j$. The definitions of η_i and F_i imply $\eta_i|_{F_i} \equiv \mathcal{O}_{F_i}$. Moreover, if we assume $\eta|_{F_i} \equiv \mathcal{O}_{F_i}$ then $h^0(F_i, \mathcal{O}_{F_i}(K_{F_i} + \eta|_{F_i})) = h^0(F_i, \mathcal{O}_{F_i}(K_{F_i})) = 3$ because the curve F_i has genus 3 by Proposition 4.2. It induces a contradiction by Lemma 4.15 (ii). \square

5. PROOF OF THEOREM 1.1

We provide the characterization of Burniat surfaces with $K^2 = 4$ and of non nodal type. We use the notations in Notations 4.1 and 4.13, and we work with Assumption 4.12. We follow the approaches in [7, 21].

Lemma 5.1 (Note Lemma 5.1 in [21]). *Let $u: S \rightarrow \mathbb{P}^1$ be a fibration such that E_4 and E_5 are contained in fibers. Then u is induced by one of the pencils $|F_i|$, $i = 1, 2, 3$.*

Proof. We argue as in the proof of Lemma 5.7 in [7]. \square

Remark 5.2. In Lemma 5.1 E_4 and E_5 are not contained in the same fiber of u because u is induced by one of $|F_i|$, $i = 1, 2, 3$.

Proof of Theorem 1.1. Let $\pi_i: Y_i \rightarrow S$ be the double cover branched over E_4 and E_5 given by the relation $2(-\eta + \eta_i) \equiv E_4 + E_5$. By Corollary 4.16 $\eta_i \neq \eta_j$ for $i \neq j$. So π_i is different from π_j . Serre duality and the formula for $q(Y)$ in Subsection 3.1 imply $q(Y_i) = h^1(S, \mathcal{O}_S(\eta - \eta_i)) = 1$ from Lemma 4.15 (iii). Let $\alpha_i: Y_i \rightarrow C_i$ be the Albanese pencil where C_i is an elliptic curve. By Proposition 3.1 there exist a fibration $h_i: S \rightarrow \mathbb{P}^1$ and a double cover $\pi'_i: C \rightarrow \mathbb{P}^1$ such that $\pi'_i \circ \alpha_i = h_i \circ \pi_i$. Since $\pi_i^{-1}(E_4)$ and $\pi_i^{-1}(E_5)$ are rational curves they are contained in fibers of α_i . So E_4 and E_5 are contained in fibers of h_i . Thus $h_i = u_{s_i}$ for some $s_i \in \{1, 2, 3\}$ by Lemma 5.1. We obtain the following commutative diagram:

$$\begin{array}{ccc} Y_i & \xrightarrow{\pi_i} & S \\ \alpha_i \downarrow & & \downarrow u_{s_i} \\ C_i & \xrightarrow{\pi'_i} & \mathbb{P}^1 \end{array}$$

By Corollary 4.16 $(-\eta + \eta_i)|_{F_i} \neq \mathcal{O}_{F_i}$. It implies that a general curve in $\pi_i^*(|F_i|)$ is connected. Hence $s_i \neq i$.

We devide the proof into six steps.

Step 1: *The fibration $u_i: S \rightarrow \mathbb{P}^1$, $i = 1, 2, 3$ has exactly two double fibers.*

It is enough to show that $u_3: S \rightarrow \mathbb{P}^1$ has at most two double fibers because u_3 already has two different double fibers, $2(E_1 + E'_2)$ and $2(E_2 + E'_1)$. Since $s_3 \neq 3$ we may consider $u_{s_3} = u_1$. Assume that u_3 has one additional double fiber $2M$ aside from $2(E_1 + E'_2)$ and $2(E_2 + E'_1)$. Then M is reduced and irreducible by Proposition 4.3 because $ME_3 = 1$ and $\varphi(M)$ is irreducible. So φ is ramified along M because the curve in the pencil $|f_3|$ supported on $\varphi(M)$ is reduced.

Let R be the ramification divisor of the bicanonical morphism $\varphi: S \rightarrow \Sigma \subset \mathbb{P}^4$. We have $\varphi^*(-K_\Sigma) \equiv 2K_S$ since the image Σ of φ is a del Pezzo surface of degree 4 in \mathbb{P}^4 (See Notations 3.3 and 4.1). It implies $R \equiv K_S + \varphi^*(-K_\Sigma) \equiv 3K_S$ by Hurwitz formula $K_S \equiv \varphi^*(K_\Sigma) + R$. Put $R_0 := \sum_{i=1}^3 (E_i + E'_i) + G_1 + G_2 + H_1 + H_2 + M$. By Assumption 4.12 φ is ramified along E_i , E'_i , G_j and H_j for $i = 1, 2, 3$ and $j = 1, 2$. It follows $R_0 \leq R$. So we get a nonzero effective divisor $E := 2(R - R_0) \equiv F_3 - E_4 - E_5$.

However, it induces a contradiction because $0 < EK_S = (F_3 - E_4 - E_5)K_S = 0$ since $K_SF_3 = 4$ by Proposition 4.2, E_4 and E_5 are (-4) -curves and K_S is ample by Proposition 4.3.

Similarly we get that u_1, u_2 each has exactly two double fibers.

Step 2: $(s_1 s_2 s_3)$ is a cyclic permutation.

Since $s_i \neq i$ we need $s_i \neq s_j$ if $i \neq j$. We verify $s_1 \neq s_2$. Otherwise, it is $s_1 = s_2 = 3$, and $\alpha_1: Y_1 \rightarrow C_1$ (resp. $\alpha_2: Y_2 \rightarrow C_2$) arises in the Stein factorization of $u_3 \circ \pi_1$ (resp. $u_3 \circ \pi_2$). We have the following commutative diagram:

$$\begin{array}{ccccc} Y_1 & \xrightarrow{\pi_1} & S & \xleftarrow{\pi_2} & Y_2 \\ \alpha_1 \downarrow & & \downarrow u_3 & & \downarrow \alpha_2 \\ C_1 & \xrightarrow{\pi'_1} & \mathbb{P}^1 & \xleftarrow{\pi'_2} & C_2 \end{array}$$

For $i = 1, 2$ Y_i coincides with the normalization of the fiber product $C_i \times_{\mathbb{P}^1} S$ since π_i factors through the natural projection $C_i \times_{\mathbb{P}^1} S \rightarrow S$ which is also of degree 2. Thus π'_1 is different from π'_2 . We denote $q_1, q_2, q_3 = u_3(E_4), q_4 = u_3(E_5)$ as the branch points of π'_1 . Then we find a branch point q_5 of π'_2 which is not branched over by π'_1 . We have the fibers over the points $q_i, i = 1, 2, 5$ of u_3 are double fibers. It is a contradiction by **Step 1**.

From now on we assume $s_1 = 2, s_2 = 3, s_3 = 1$, and for each $i \in \{1, 2, 3\}$ the fibration u_i has exactly two double fibers.

Step 3: $\varphi^*(g_3)$ and $\varphi^*(h_3)$ are not reduced.

We have the following commutative diagram:

$$\begin{array}{ccc} Y_2 & \xrightarrow{\pi_2} & S \\ \alpha_2 \downarrow & & \downarrow u_3 \\ C_2 & \xrightarrow{\pi'_2} & \mathbb{P}^1 \end{array}$$

Let W be $C_2 \times_{\mathbb{P}^1} S$, and let $p: W \rightarrow S$ be the natural projection which is a double cover. Assume that $G_3 := \varphi^*(g_3)$ (resp. $H_3 := \varphi^*(h_3)$) is reduced. Since $\pi'_2: C_2 \rightarrow \mathbb{P}^1$ is branched over the point $u_3(G_3) = u_3(E_4)$ (resp. $u_3(H_3) = u_3(E_5)$), the map p is branched over G_3 (resp. H_3). Thus W is normal along $p^{-1}(G_3)$ (resp. $p^{-1}(H_3)$). The map $\pi_2: Y_2 \rightarrow S$ is also branched over G_3 (resp. H_3) because Y_2 is the normalization of W . It is a contradiction because the branch locus of π_2 is $E_4 \cup E_5$.

Step 4: A general element $F_i \in |F_i|$ is hyperelliptic for each $i \in \{1, 2, 3\}$.

We verify that a general fiber $F_2 \in |F_2|$ is hyperelliptic. Since the pull-back $\pi_1^*(F_2)$ (resp. $\pi_1^*(F_3)$) is disconnected, we may consider $\pi_1^*(F_2) = \hat{F}_2 + \hat{F}_2'$ (resp. $\pi_1^*(F_3) = \hat{F}_3 + \hat{F}_3'$) where the two components are disjoint. Then we get $\hat{F}_2\hat{F}_3 = 2$ by $F_2F_3 = 4$. Let $p \circ h: Y_1 \rightarrow C \rightarrow \mathbb{P}^1$ be the Stein factorization of $u_3 \circ \pi_1: Y_1 \rightarrow \mathbb{P}^1$. Since \hat{F}_3 is a fiber of $h: Y_1 \rightarrow C$, the restriction map $h|_{\hat{F}_2}: \hat{F}_2 \rightarrow C$ is a 2-to-1 map by $\hat{F}_2\hat{F}_3 = 2$. Moreover, C is rational because $h: Y_1 \rightarrow C$ is not Albanese map and $q(Y_1) = 1$. Thus \hat{F}_2 is hyperelliptic, and so is F_2 .

Step 5: $\varphi: S \rightarrow \Sigma$ is a Galois cover with the Galois group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

For each $i \in \{1, 2, 3\}$, let γ_i be the involution on S induced by the involution on the general fiber F_i . Since S is minimal, the maps γ_i are regular maps. So the maps γ_i belong to G by Proposition 4.14. Now it suffices $\gamma_i \neq \gamma_j$ if $i \neq j$. We show $\gamma_2 \neq \gamma_3$. Consider the lifted involution $\hat{\gamma}_2: Y_1 \rightarrow Y_1$. The restriction

of α_1 identifies $\hat{F}_3/\hat{\gamma}_2$ with C_1 by the construction in Step 4. Thus we obtain $p_g(\hat{F}_3/\hat{\gamma}_2) = 1$, but $\hat{F}_3/\hat{\gamma}_3 \cong \mathbb{P}^1$. It means that $\gamma_2 \neq \gamma_3$.

Step 6: S is a Burniat surface.

Denote by B be the branch divisor of φ . Then we get

$$-3K_\Sigma \equiv B \geq \sum_{i=1}^3 (e_i + e'_i + g_i + h_i) \equiv -3K_\Sigma,$$

thus $B = \sum_{i=1}^3 (e_i + e'_i + g_i + h_i)$. Denote B_i as the image of the divisorial part of the fixed locus of γ_i . We have $B = B_1 + B_2 + B_3$. By **Step 4** we obtain $B_i = e_i + e'_i + g_{i+1} + h_{i+1}$ for each $i \in \{1, 2, 3\}$, where g_4 (*resp.* h_4) denotes g_1 (*resp.* h_1).

The theorem is proved with all steps. \square

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